B.Sc. Part I (Hons) 1st Paper Trigonometry

Summation of series (Contd.)

<u>1</u> Sum the series

$$\frac{2}{1\cdot 3}\sin 2x - \frac{4}{3\cdot 5}\sin 4x + \frac{6}{5\cdot 7}\sin 6x + \cdots,$$

where $0 < x < \frac{\pi}{2}$.

<u>Soln</u> The given series

$$= \frac{1}{2}\left[\frac{4}{1\cdot 3}\sin 2x - \frac{8}{3\cdot 5}\sin 4x + \frac{12}{5\cdot 7}\sin 6x + \cdots\right]$$

$$= \frac{1}{2}\left[\frac{1+3}{1\cdot 3}\sin 2x - \frac{3+5}{3\cdot 5}\sin 4x + \frac{5+7}{5\cdot 7}\sin 6x + \cdots\right]$$

$$= \frac{1}{2}\left[\left(1+\frac{1}{3}\right)\sin 2x - \left(\frac{1}{3}+\frac{1}{5}\right)\sin 4x + \left(\frac{1}{5}+\frac{1}{7}\right)\sin 6x + \cdots\right]$$

$$= \frac{1}{2}\left[\left(\sin 2x - \frac{1}{3}\sin 4x + \frac{1}{5}\sin 6x - \cdots\right) + \left(\frac{1}{3}\sin 2x - \frac{1}{5}\sin 4x + \frac{1}{7}\sin 6x - \cdots\right)\right]$$

$$= \frac{1}{2}\left[\left(\sin 2x - \frac{1}{3}\sin 4x + \frac{1}{5}\sin 6x - \cdots\right) - \left(-\frac{1}{3}\sin 2x + \frac{1}{5}\sin 4x - \frac{1}{7}\sin 6x + \cdots\right)\right]$$

Let

$$\sin 2x - \frac{1}{3}\sin 4x + \frac{1}{5}\sin 6x - \cdots = S_1$$

$$-\frac{1}{3}\sin 2x + \frac{1}{5}\sin 4x - \frac{1}{7}\sin 6x + \cdots = S_2 \quad (2)$$

So, the given series

$$= \frac{1}{2}(S_1 - S_2) \quad \text{———} \quad (3)$$

Now, consider S_1 where

$$S_1 = \sin 2x - \frac{1}{3}\sin 4x + \frac{1}{5}\sin 6x \cdots$$

Let $C_1 = \cos 2x - \frac{1}{3}\cos 4x + \frac{1}{5}\cos 6x \cdots$

$$\Rightarrow C_1 + iS_1 = (\cos 2x + i\sin 2x) - \frac{1}{3}(\cos 4x + i\sin 4x)$$

$$+ \frac{1}{5}(\cos 6x + i\sin 6x) \cdots$$

$$= e^{2ix} - \frac{1}{3}e^{i4x} + \frac{1}{5}e^{6ix} \cdots$$

$$= e^{ix}\left[e^{ix} - \frac{1}{3}e^{3ix} + \frac{1}{5}e^{5ix} \cdots\right]$$

$$\Rightarrow C_1 + iS_1 = e^{ix}\tan^{-1}(e^{ix}) \quad \text{———} \quad (4)$$

Again, from (2)

$$S_2 = \cancel{1} - \frac{1}{3}\sin 2x + \frac{1}{5}\sin 4x - \frac{1}{7}\sin 6x -$$

Let

$$C_2 = 1 - \frac{1}{3}\cos 2x + \frac{1}{5}\cos 4x - \frac{1}{7}\cos 6x + -$$

$$\Rightarrow C_2 + iS_2 = 1 - \frac{1}{3}(\cos 2x + i\sin 2x) + \frac{1}{5}(\cos 4x + i\sin$$

$$- \frac{1}{7}(\cos 6x + i\sin 6x) + \cdots$$

$$\Rightarrow C_2 + iS_2 = 1 - \frac{e^{2ix}}{3} + \frac{1}{5}e^{4ix} - \frac{1}{7}e^{6ix} + -$$

$$= \frac{1}{e^{ix}}\left[e^{ix} - \frac{e^{3ix}}{3} + \frac{1}{5}e^{5ix} - \frac{1}{7}e^{7ix} + -\right]$$

$$\Rightarrow C_2 + iS_2 = e^{-ix} \cdot \tan^{-1}(e^{ix})$$

Now $(C_1 + iS_1) - (C_2 + iS_2) = (C_1 - C_2) + i(S_1 - S_2)$

So, $S_1 - S_2$ is the imaginary part of $(C_1 + iS_1) - (C_2 + iS_2)$.

i.e. $S_1 - S_2$ is the I.P. of $e^{ix}\tan^{-1}(e^{ix}) - e^{-ix}\tan^{-1}(e^{ix})$

$\Rightarrow S_1 - S_2$ is the imaginary part of

$\left(e^{ix} - e^{-ix}\right) \tan^{-1}\left(e^{ix}\right) = 2i \sin x \tan^{-1} e^{ix}$. — (5)

Let $p + iq = \tan^{-1}\left(e^{ix}\right) \Rightarrow \cancel{\tan^{-1}(\cos x + i \sin x)}$

$\Rightarrow p - iq = \tan^{-1}\left(e^{-ix}\right)$

Adding the above two, we get -

$\Rightarrow \cancel{p}\ 2p = \tan^{-1}\left(e^{ix}\right) + \tan^{-1}\left(e^{-ix}\right)$

$\Rightarrow 2p = \tan^{-1} \dfrac{e^{ix} + e^{-ix}}{1 - e^{ix} \cdot e^{-ix}} = \tan^{-1}\left(\dfrac{2 \cos x}{1 - 1}\right)$

$= \tan^{-1} \infty = \dfrac{\pi}{2}$

$\Rightarrow p = \dfrac{\pi}{4}$

So, from (5)

$S_1 - S_2 = I$ P of $2i \sin x \tan^{-1} e^{ix} = 2i \sin x \cdot (p + iq)$

$\Rightarrow S_1 - S_2 = 2p \sin x$ but $p = \dfrac{\pi}{4}$

$\Rightarrow S_1 - S_2 = 2 \times \dfrac{\pi}{4} \sin x = \dfrac{\pi}{2} \sin x$

Hence the sum $= \dfrac{1}{2}\left(\dfrac{\pi}{2} \sin x\right) = \dfrac{\pi}{4} \sin x$.